Euler Equation Approach for Emerging-Market Macro Models

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This Draft: April, 2013

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Abstract

In this paper, our focus is given on how to obtain numerical solutions to emerging-market DSGE models with occasionally binding constraints by using the Euler equation, rather than using value functions of households. Our main point of this paper is that the Euler-equation approach works in a fast and simple way for a variety of recent emerging-market macro models. An important reason behind this point is that it is relatively easy to pin down the functional form of aggregate equilibrium conditions in these models. The time-iteration method is applied to Euler equations of a small open-economy with overborrowings. We discuss how to use the Euler equation approach to recent models of sovereign debt and show that the presence of the Laffer-curve of debt-revenues leads us to use piecewise parameterized-expectations approach.

1 Introduction

Emerging economies have experienced economic crises more frequently than advanced countries. In particular, the recent financial integration of emerging market economies into international financial markets has led them to experience far more volatile fluctuations in goods and financial markets at the same time. For example, strong capital inflows during good times create the issue of overborrowings and strong capital outflows during bad times often trigger systemic financial crises. In this vein, academic researchers and policy makers have addressed the issue of how to moderate boom-bust cycles in foreign capital flows.

A series of recent academic works have formulated this feature of emerging economies in small-scale DSGE models with occasionally binding constraints. Specifically, in theses models, a financial crisis is identified as a time when a constraint binds, while this constraint does not necessarily binds at all times. In order to solve emerging-market DSGE models with occasionally binding constraints, recent papers have utilized the discrete state space technique that discretizes the stochastic process for real output and thus restricts the government to choose the optimal borrowing level from a discrete-set of points. In doing so, they use value functions of households in characterizing equilibrium conditions to be used in the computation of numerical solutions. In this paper, our focus is given on the Euler equation approach for emerging-market DSGE models with occasionally binding constraints, rather than using value functions of households.

Our main point of this paper is that the Euler-equation approach works in a fast and simple way for a variety of recent emerging-market macro models. The key reason behind this point is the absence of the potential complexity associated with the aggregation of individual decision rules in these models, while they use the concept of recursive competitive equilibrium. Hence, it is relatively easy to pin down the functional form of aggregate equilibrium conditions.

In this paper, when we solve Euler equations numerically for emerging-market macroeconomic models, we exploits the idea of Rendahl (2006) that uses the well-known Lagrange multiplier method in solving dynamic models with inequality constraints. The use of Lagrange multiplier method facilitates the adoption of the time-iteration method, which uses Euler equations to find current-period's policy functions given next-period's policy functions.

In our first example, this time-iteration method is applied to Euler equations of a small open-economy with tradable and non-tradable goods sectors analyzed in Bianchi (2011). It is then shown that our Euler-equation approach can help replicate numerical solutions based on value-function iteration. It would be also worthwhile to discuss two features of our time-iteration method, compared with value function iteration. First, the time-iteration method exploits only aggregate equilibrium conditions. So it is sufficient to use only the law of motion for the aggregate debt at

a symmetric equilibrium. In contrast, the computation of a recursive competitive equilibrium (by using value function iteration) requires two different laws of motion for aggregate and individual debts. The planning problem reflects the impact of future Lagrange multiplier on the Euler equation, whereas the decentralized equilibrium does not allow for such an effect.

In the second example, we discuss how to use the Euler equation approach to recent models of sovereign debt such as Arellano (2008) and Hatchondo, Martinez, and Sapriza (2010). In particular, we will show that the presence of the laffer-curve of debt-revenues leads us to use piecewise parameterized-expectations approach, which means different parameterizations of conditional expectations for different ranges of outstanding debts with some smooth-pasting conditions.

The organization of this paper can be summarized as follows. In sections 2 and 3, we discuss how to use time-iteration and parameterized-expectations methods to the model of overborrowings that generates equilibrium conditions with occasionally binding constraints. In section 4, we use a piecewise parameterized-expectations approach to solve a model of sovereign debt. Section 5 concludes.

2 Time-Iteration Method for Equilibrium Conditions with Occasionally Binding Constraints

We begin with a dynamic stochastic general equilibrium model whose equilibrium conditions can be partitioned into three sets of dynamic equations: first-order conditions of optimization problems, equality constraints and occasionally binding constraints. Let us denote the vector at period t of endogenous non-state variables by y_t and the vector at period t of state variables by x_t, following Hintermaier and Koeniger (2010) and Anderson (2012).

The first-order conditions including Lagrange multipliers can be written as follows:

$$E_t[\mathcal{F}(y_t, x_t, \mu_t, y_{t+1}, x_{t+1}, \mu_{t+1})] = 0$$

where μ_t is the Lagrange multiplier at period t and $E_t[z_{t+1}]$ is the conditional expectation at period t of a vector z_{t+1}. The equality constraints are

$$E_t[\mathcal{Q}(y_t, x_t, y_{t+1}, x_{t+1})] = 0.$$

The occasionally binding constraints are summarized in the following equation:

$$E_t[\mathcal{O}(y_t, x_t, y_{t+1}, x_{t+1})] = 0.$$

The complementary slackness conditions are given by

$$\mu_t E_t[\mathcal{O}(y_t, x_t, y_{t+1}, x_{t+1})] = 0$$

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with $\mu_t \geq 0$.

The time iteration method is defined as an iterative procedure that uses a set of optimization conditions in order to find a current-period's policy function taking as given a next-period's policy function. In order to see how this method works, let us begin with a policy function $y_t = h^{(n)}(x_t)$ at a round $n = 0, 1, 2, \cdots$. The substitution of $y_t = h^{(n)}(x_t)$ into equality constraints leads us to have an implicit function for state variables at current and next periods:

$$\mathcal{Q}(h^{(n)}(x_t), x_t, h^{(n)}(x_{t+1}), x_{t+1}) = 0.$$

Solving this equation for the state vector at the next-period yields an evolutionary equation for the state-vector: $x_{t+1} = g^{(n)}(x_t)$. We then put $y_t = h^{(n)}(x_t)$ and $x_{t+1} = g^{(n)}(x_t)$ into the set of first-order conditions, so that we have an equilibrium relation between non-state and state variables as follows:

$$\mathcal{F}(y_t, x_t, \mu_t, h^{(n)}(g^{(n)}(x_t)), g^{(n)}(x_t), l(g^{(n)}(x_t))) = 0.$$

In addition, the occasionally binding constraint can be written as follows:

$$\mathcal{O}(y, x_t, h^{(n)}(g^{(n)}(x_t)), g^{(n)}(x_t)) = 0.$$

By solving these two sets of equations at the same time, we can update policy functions for non-state variables and Lagrange multipliers: $y_t = h^{(n+1)}(x_t)$ and $\mu_t = l^{(n+1)}(x_t)$.

2.1 Model Specification

We now highlight key features of a simple dynamic model that addresses the issue of overborrowings. Following Bianchi (2011), we consider a small open-economy with a tradable goods sector and a non-tradable goods sector.

- The preferences of households at period 0 are represented by $E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)]$ where $0 < \beta < 1$ is the time discount factor and c_t is the consumption at period t.

- The consumption basket c_t is an Armington-type CES aggregator with elasticity of substitution $1/(1+\eta)$ between tradable and non-tradable consumption goods:

$$c_t = (\omega(c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta})^{-1/\eta}$$

where $\eta > 1$ and $\omega \in (0, 1)$.

- The vector of endowments $y \equiv (y^T, y^N)$ follows a first-order Markov process. These endowment shocks are the only source of uncertainty in the model.

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Table 1: Decentralized Equilibrium versus Planner's Optimality Conditions

$\lambda_t = MU_t$	$\lambda_t^{sp} = MU_t + \mu_t^{sp}\Phi_t$
$p_t^N = \frac{\omega}{1-\omega}(c_t^T/y_t^N)^{1+\eta}$	$p_t^N = \frac{\omega}{1-\omega}(c_t^T/y_t^N)^{1+\eta}$
$\lambda_t = \beta(1+r)E_t[\lambda_{t+1}] + \mu_t$	$\lambda_t^{sp} = \beta(1+r)E_t[\lambda_{t+1}^{sp}] + \mu_t^{sp}$
$c_t^T = y_t^T + b_{t+1} - (1+r)b_t$	$c_t^T = y_t^T + b_{t+1}^{sp} - (1+r)b_t^{sp}$
$b_{t+1} \leq \kappa(p_t^N y_t^N + y_t^T)$	$b_{t+1}^{sp} \leq \kappa(p_t^N y_t^N + y_t^T)$

Note: The left column describes decentralized equilibrium conditions and the right column contains the social planner's optimality conditions. The equilibrium conditions in this table are taken from Bianchi (2011). MU_t is the marginal utility of consumption. $\Phi_t = \kappa p_t^N c_t^N/((1+\eta)c_t^T)$.

- The menu of foreign assets available is restricted to a one-period non-state contingent bond denominated in units of tradable that pays a fixed interest rate, r, determined exogenously in the world market.

- The budget constraint of an individual household can be written as

$$b_t(1+r) + c_t^T + p_t^N c_t^N = b_{t+1} + y_t^T + p_t^N y_t^N$$

where p_t^N is the relative price of non-tradable consumption goods. In this budget constraint and b_t is the real amount of debt (not the real value of asset holdings).

- The collateral constraint is

$$b_{t+1} \leq \kappa(y_t^T + p_t^N y_t^N)$$

It would be worthwhile to discuss some features associated with the social planner's problem. The optimality conditions of the social planner's problem are summarized in Table 1. Since the planner's choice internalizes the effect of tradable consumption goods on the relative price of non-tradable consumption goods, the lagrange multiplier of the social resource constraint is affected by the marginal utility of consumption and the lagrange multiplier of the collateral constraint, as shown in the left column of Table 1. As a result, the planning problem reflects the impact of future lagrange multiplier on the euler equation, whereas the decentralized equilibrium does not allow for such an effect.

In sum, we can emphasize two features of our time-iteration method.

Figure 1: Decentralized Equilibrium: Debt Policy Function
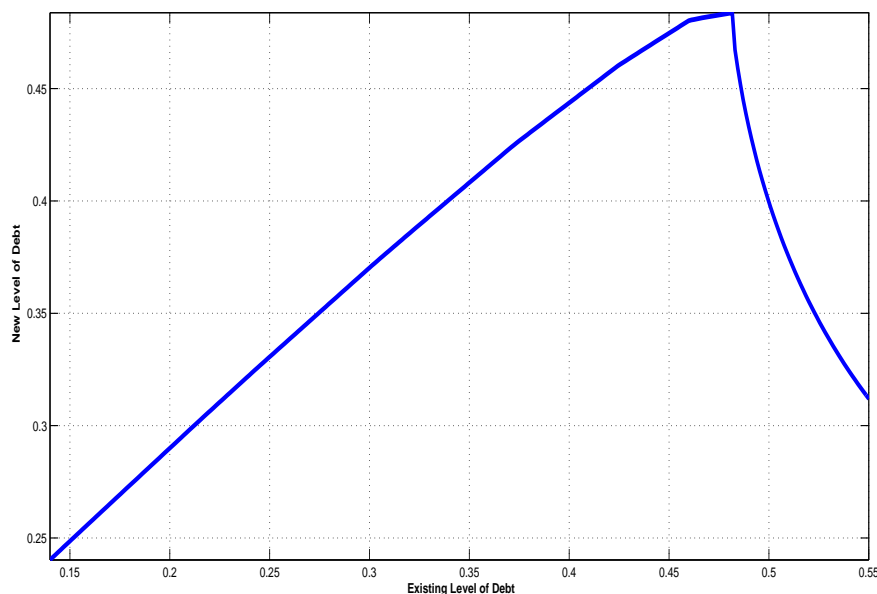
1. This time-iteration method exploits only aggregate equilibrium conditions. So it is sufficient to use the aggregate law of motion for the debt. In contrast, the implementation of the recursive competitive equilibrium requires two different laws of motion for debts.

2. The planning problem reflects the impact of future Lagrange multiplier on the Euler equation, whereas the decentralized equilibrium does not allow for such an effect.

Figure 1 plots the debt policy function of the decentralized equilibrium when the realized value of tradable output is relatively low. In this figure, we can find that although the aggregate debt rises with the existing debt, it begins to decline starting from the point at which the collateral constraint is binding. In our calibration, the collateral constraint is not binding when the tradable output is sufficiently large.

2.2 Value Function Method

The discretization of value functions has been widely used in many emerging-economies macro models in order to solve dynamic models with occasionally binding credit constraints. A typical example is to solve a decentralized recursive competitive equilibrium. Specifically, the definition of a decentralized recursive competitive equilibrium can exemplified by using the model Javier Bianchi (2011). A recursive decentralized competitive equilibrium for the small open economy is a pricing

function $p^N(B; y^T)$; a perceived law of motion $\Gamma(B; y^T)$ and decision rules $\{\hat{b}(b, B, y^T), \hat{c}^T(b, B, y^T),$ $\hat{c}^N(b, B, y^T)\}$ such that the following conditions hold:

- Household optimization: taking as given $p^N(B, y^T)$ and $\Gamma(B; y^T)$, $\{\hat{b}(b, B, y^T), \hat{c}^T(b, B, y^T),$ $\hat{c}^N(b, B, y^T)\}$ solve the solve the optimization problem of the household.

- Rational expectation condition: the perceived law of motion is consistent with the actual law of motion: $\Gamma(B, y^T) = \hat{b}(B, B, y^T)$.

- Markets clear: $y^N = c^N(B; B; y^T)$ and $\Gamma(B; y^T) + c^T = y^T + B(1+r)$.

Specifically, the value function of a representative household can be written as

$$v(b, B, y) = \max_{b', c^T, c^N} \{u(c(c^T, c^N) + \beta E[V(b', B', y')]\}$$

subject to $B' = \Gamma(B, y)$, $b' \leq \kappa(p^N(B, y)y^N + y^T)$, and

$$b' + p^N(B, y)c^N + c^T = y^T + b(1+r) + p^N(B, y)y^N.$$

The algorithm employed in the paper of Binach (2011) to solve for the recursive competitive equilibrium can be summarized as follows.

1. Choose a set of parameter values $\{\beta, \omega, \kappa\}$.

2. Create a set of grid points for the economy's asset of size N_B and the state space N_S.

3. Begin with an initial guess on a law of motion for aggregate foreign assets $B' = H(B, y^T)$ at each point of the state space (with $Y^N = C^N$).

4. Solve for households' policy functions $\{\hat{b}(b, B, y^T), \hat{C}^T(b, B, y^T), \hat{C}^N(b, B, y^T)\}$ via value-function iteration.

5. Using the decision rules in the previous step, calculate the implied transition of aggregate bonds $\hat{b}(B; B; y^T)$ and evaluate the previous guess. If $\sup ||\hat{b}(B, B, y^T) - \Gamma(B, y^T)|| < \epsilon$, then the recursive competitive equilibrium is found.

3 Parameterized Expectations Approach: Models of Overborrowings

We now briefly discuss how a parameterized expectation method can be used to solve the model of overborrowings specified above.

Table 2: Decentralized Equilibrium Conditions for Parameterized-Expectations Method

$\mu_t > 0$	$\mu_t = 0$
$c_t^T = \theta_y y_t^T - \theta_b b_t$	$c_t^T = y_t^T + b(y_t^T, b_t) \text{ - } (1+r)b_t$
$b_{t+1} = \beta_y y_t^T - \beta_b b_t$	$b_{t+1} = b(y_t^T, b_t)$
$c_t = \omega(c_t^T)^\eta (y_t^N)^{1-\eta}$	$c_t = \omega(c_t^T)^\eta (y_t^N)^{1-\eta}$
$\lambda_t = \omega\eta c_t^{-\sigma}(y_t^N/c_t^T)^{1-\eta}$	$\lambda_t = \omega\eta c_t^{-\sigma}(y_t^N/c_t^T)^{1-\eta}$
$\mu_t = \lambda_t \text{ - } \beta(1+r)E_t[\lambda_{t+1}]$	$\lambda_t = \beta(1+r)E_t[\lambda_{t+1}]$

Note: The left column describes decentralized equilibrium conditions when the collateral constraint is binding. The right column describes decentralized equilibrium conditions when the collateral constraint is not binding. $\theta_y = (1+\kappa)/(1-\kappa(1-\eta))$, $\theta_b = (1+r)/(1-\kappa(1-\eta))$, $\beta_y = \kappa(1-\eta)\theta_y$, and $\beta_b = \kappa\theta_b$.

The parametrization of the conditional expectations of the Euler equation is made by defining a new function as follows:

$$\exp(e(\tilde{b}_t, \tilde{y}_t^T)) \equiv (1+r)E_t[\lambda_{t+1}].$$

Given this parameterizations, the Euler equation can be rewritten as

$$\mu_t = \lambda_t - \beta \exp(e(\tilde{b}_t, \tilde{y}_t^T)).$$

When the collateral constraint is binding, we solve the household's flow budget and collateral constraints for consumption and debt policy functions. In this case, the household's budget and collateral constraints can be rewritten as follows:

$$b_t(1+r) + c_t^T = (1+\kappa)y_t^T + \frac{\omega}{1-\omega}(c_t^T)^{1+\eta}(y_t^N)^{-\eta}$$

$$b_{t+1} = \kappa y_t^T + \frac{\kappa\omega}{1-\omega}(c_t^T)^{1+\eta}(y_t^T)^{-\eta}$$

In addition, when the collateral constraint is not binding, the Euler equation can be written as follows:

$$\omega(c_t^T)^{-\sigma}[\omega + (1-\omega)(y_t^N)^{-\eta}(c_t^T)^\eta] = \beta \exp(e(\tilde{b}_t, y_t^T)).$$

We can solve this equation to obtain the consumption function: $c_t^T = c^T(\tilde{b}_t, y_t^T)$. In this case, the debt policy function is

$$b_{t+1} = y_t^T - (1+r)\exp(\tilde{b}_t) - c^T(\tilde{b}_t, y_t^T).$$

Table 3: Social Planner's Optimality Conditions for Parameterized-Expectations Method

$\mu_t > 0$	$\mu_t = 0$
$c_t^T = \theta_y y_t^T - \theta_b b_t$	$c_t^T = y_t^T + b(y_t^T, b_t) - (1+r)b_t$
$b_{t+1} = \beta_y y_t^T - \beta_b b_t$	$b_{t+1} = b(y_t^T, b_t)$
$c_t = \omega(c_t^T)^\eta (y_t^N)^{1-\eta}$	$c_t = \omega(c_t^T)^\eta (y_t^N)^{1-\eta}$
$\lambda_t = \omega\eta c_t^{-\sigma}(y_t^N/c_t^T)^{1-\eta} - \mu_t\Psi_t$	$\lambda_t = \omega\eta c_t^{-\sigma}(y_t^N/c_t^T)^{1-\eta}$
$\mu_t = \lambda_t - \beta(1+r)E_t[\lambda_{t+1}]$	$\lambda_t = \beta(1+r)E_t[\lambda_{t+1}]$

Note: The left column describes planner's optimality conditions when the collateral constraint is binding. The right column describes planner's optimality conditions when the collateral constraint is not binding. $\theta_y = (1+\kappa)/(1-\kappa(1-\eta))$, $\theta_b = (1+r)/(1-\kappa(1-\eta))$, $\beta_y = \kappa(1-\eta)\theta_y$, and $\beta_b = \kappa\theta_b$.

Once we obtain consumption and debt policy functions, we can use them to compute the conditional expectation in the euler equation. We thus now discuss how to find an approximate function for the conditional expectation function. Specifically, while the residual function is defined as the difference between the conditional expectation function in the euler equation and its approximation, a specific function form of the approximate function can be obtained by minimizing the residual function. The following procedure is used to obtain a specific functional form of the conditional expectation function in the Euler equation. The conditional expectation is given by $\int_{\tilde{y}'} m(g(\tilde{b}, \tilde{y}), \tilde{y}')f(\tilde{y}', \tilde{y})d\tilde{y}'$ where $g(\tilde{b}, \tilde{y})$ is the debt policy function and $\lambda(\tilde{b}, \tilde{y}) = m(\tilde{b}, \tilde{y})$ is the lagrange multiplier of the household's flow budget constraint. The residual function is then defined as $R^{\text{pea}}(\tilde{b}, \tilde{y}; e) = e(\tilde{b}, \tilde{y}) - (1+r)\int_{\tilde{y}'} m(g(\tilde{b}, \tilde{y}), \tilde{y}')f(\tilde{y}', \tilde{y})d\tilde{y}'$. The policy function for the conditional expectation is approximated by the following Chebyshev polynomials: $e(\tilde{b}, y_i) \approx a'_{y_i}T(\varphi(\tilde{b}))$, where $T(x) = [T_0(x), T_1(x), \cdots, T_{N-1}(x)]$ and $\varphi(\tilde{b}) = 2(\tilde{b} - \tilde{b}_{\min})/(\tilde{b}_{\max} - \tilde{b}_{\min}) - 1$. An example of two states for the tradable output (y^h and y^l), the conditional expectations are parameterized by using the following functions: $e(\tilde{b}, y_h) \approx a'_{y_h}T(\varphi(\tilde{b}))$ and $e(\tilde{b}, y_h) \approx a'_{y_l}T(\varphi(\tilde{b}))$.

Figure 2 plots debt policy functions obtained by using the parameterized-expectations method. In this figure, we contrast the debt policy function of the decentralized equilibrium with that of the planning problem. Figure 2 illustrates that the optimal amount of debt is lower than that of the private sector's equilibrium. The difference between these two debt functions reflects the presence of over-borrowings that arises because of the externality at the collateral constraint.

Figure 2: Decentralized Equilibrium versus Planner's Problem: Debt Policy Function
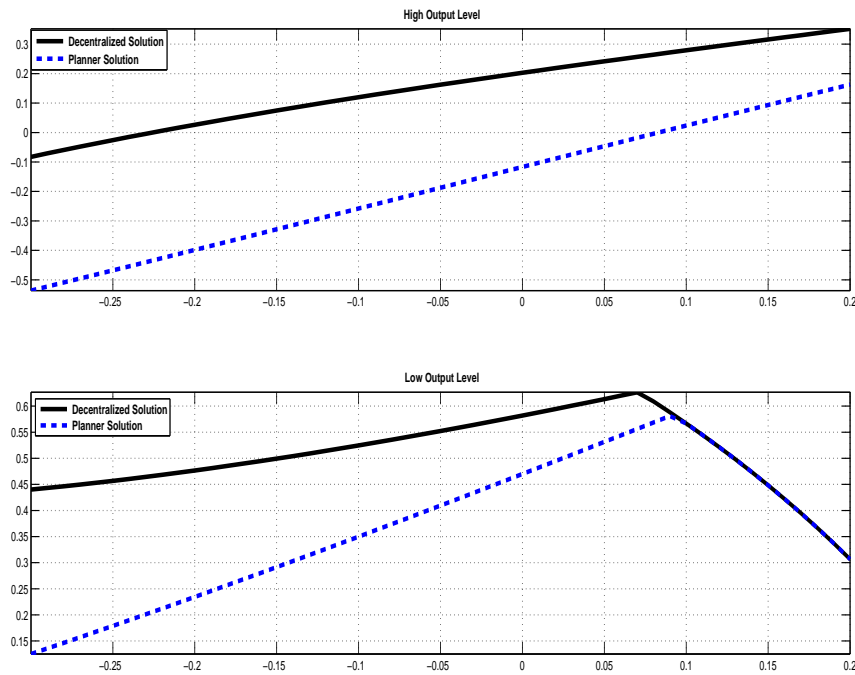
Note: The solid line represents the debt policy function at the decentralized equilibrium. The solid line corresponds to the debt policy function of the planning problem. The horizontal axis represents the level of the existing debt carried from the previous period.

4   Piecewise Parameterized-Expectations Approach: Models of Sovereign Debt

In this section, we discuss both value-function and euler-equation methods that can be used to obtain numerical solutions to a simple dynamic model of sovereign debt. In particular, we will show that the laffer-curve of debt-revenue leads us to use piecewise parameterized-expectations approach when we implement the euler equation approach.

4.1   Model Specification

We now highlight key features of a simple dynamic model that has been widely used for various issues of sovereign debt. Following Arellano (2008), we consider a small open economy whose real output is exogenously determined in a random fashion. The government issues bonds and trade them with risk-neutral foreign investors. The government can choose to default on its debt at any time. When the government defaults, its country is excluded temporarily out of international

financial market.

- The preferences of households at period 0 are represented by $E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)]$ where $0 < \beta < 1$ is the time discount factor and c_t is the consumption at period t.

- An individual household' real income evolves over time according to a transition density function: $f(y', y)$ where y' is next-period output and y is current-period output.

- The government trades bonds with risk neutral competitive foreign creditors.

- Debt contracts are not enforceable and the government can choose to default on its debt at any time. As a result, the resource constraint for the small open economy is $c = y - b + q(b', y)b'$ when the government chooses to repay its debt and consumption equals output: $c = y^{\text{def}}$ when the government chooses to default. In this social resource constraint, y is real income, b is the beginning-of-period debt that is carried over from the previous period, and b' is the end-of-period debt that should be redeemed at the next period.

- If the government defaults, it is assumed to be temporarily excluded from international intertemporal trading and to incur direct output costs.

- Foreign creditors have access to an international credit market in which they can borrow or lend as much as needed at a constant international rate $r > 0$. In each period, lenders choose loans b' to maximize expected profits ϕ, taking prices given: $\phi = qb' - (1-d)b'/(1+r)$, where d is the probability of default.

- The price of each bond available to the government reflects the likelihood of default events. Specifically, the equilibrium price accounts for the risk of default: $q = (1-d)/(1+r)$ for some positive debt levels ($b' < 0$) and $q = 1/(1+r)$ for negative levels of foreign debt ($b' \geq 0$) because the probability of default is zero.

By using value functions of households, we can formulate government's decision on default as follows. To the extent which the government chooses to repay its debt, the value function of households can be written as

$$v^o(b, y) = \max_{b'}\{u(y + q(b', y)b' - b) + \beta \int_{y'} v^o(b', y')f(y', y)dy'\}$$

where $v^o(b, y)$ represents the value function for the government that does not choose to default and the value function of households under default is

$$v^d(y) = u(y^{\text{def}}) + \beta \int_{y'} (\theta v^o(0, y') + (1 - \theta)v^d(y'))f(y', y)dy'.$$

4.2 Value Function Method

Arellano (2008) used a value-function method to solve her model. According to the appendix of her paper, the following algorithm is used to solve the model:

1. Start with some guess for the parameters to be calibrated: b, θ, and $\hat{y}$ and a discretized state space for assets consisting of a grid of 200 points equally spaced.

2. Start with a guess for the bond price schedule such that $q^{(0)}(b, y) = (1 - d)/(1 + r)$ for all b' and y.

3. Given the bond price schedule, solve the optimal policy functions for consumption $c(b, y)$, asset holdings $b'(b, y)$, repayment sets $A(b)$, and default sets $D(b)$ via value function iteration. For each iteration of the value function, we need to compute the value of default which is endogenous because it depends on the value of the contract at $b = 0$. We iterate on the value function until convergence for a given $q^{(0)}$.

4. Using default sets and repayment sets, compute new bond price schedule $q^{(1)}(b, y)$ such that lenders break even and compare it to the bond price schedule of the previous iteration: $q^{(0)}(b, y)$. If a convergence criterion is met, $\max\{q^{(0)}(b, y) - q^{(1)}(b, y)\} < \epsilon$, then move to the next step. Otherwise, update the bond price schedule and go back to step 3.

5. Compute business cycles statistics from 100 samples of data containing a default. If the model business cycles match the data we stop; otherwise we adjust parameters and grid, and go to step 2.

4.3 Characterization of Equilibrium Conditions

In this section, we begin with the debt-revenue function $F(b', y) = q(b', y)b'$. Considering the shape of the debt-revenue function, we can find that there are three distinct regions for b'. In the first region, the price of bond is the inverse of the risk-free interest rate at the international capital market because there is no perceived risk of default. In the second region, the price of bond falls as the amount of debt rises, reflecting increased possibility of default. The third region corresponds to the default period when there is no access to the international financial market. In sum, there is a threshold level of outstanding debt (denoted by $\tilde{b}'$) that determines whether or not the price of bond reflects the possibility of default.

In this model, we assume that $q(b', y) = 1/(1 + r)$ holds for only a range of $(\underline{b}', \tilde{b}')$. Hence, the bond price reflects the default risk when the outstanding debt level exceeds $\tilde{b}'$. In order to take into account the Laffer-curve for the aggregate debt-revenue, we allow for an interval constraint for the

outstanding debt as follows

$$1_{[\underline{b}' \quad \tilde{b}']}b' \geq 0$$

where $\underline{b}'$ is the minimum of the outstanding debt.

Let us denote the value function for the government that does not choose to default by $v^o(B, y)$. The optimization problem of the government can be written as

$$v^o(b, y) = \max_{b'}\{u(y - q(b', y)b' + b) + \beta \int_{y'} v^o(b', y')f(y', y)dy'\}$$

subject to a sequence of constraints

$$v^o(b, y) \geq v^d(y)$$

$$1_{[\underline{b}' \quad \tilde{b}']}b' \geq 0$$

where the value function of households under default is

$$v^d(y) = u(y^{\text{def}}) + \beta \int_{y'} (\theta v^o(0, y') + (1 - \theta)v^d(y'))f(y', y)dy'.$$

The Euler equation can be written as

$$u'(c_t)F_1(b_{t+1}, y) = \beta E_t[\frac{u'(c_{t+1})}{1 + \mu_{t+1}}] - \gamma_t$$

where μ_t is the Lagrange multiplier of the value-function constraint at period t and γ_t is the Lagrange multiplier at period t for the interval constraint. The Lagrange multiplier of the value-function constraint is positive ($\mu_t > 0$) when the government is indifferent between default and non-default options $v^o(b_t, y_t) = v^d(y_t)$. The reason why we introduce this interval constraint is to allow for the negative derivative of the debt-revenue function nearby the default region.

Moreover, because of the assumption of risk-neutral investors at the international financial market, the debt-revenue function is given by

$$F(b_{t+1}, y_t)/b_{t+1} = \{ \begin{array}{lllllll} 1/(1 + r) & \text{if} & \underline{b}_{t+1} & \leq & b_{t+1} & \leq & \tilde{b}_{t+1} \\ (1 + r)^{-1}E_t[1/(1 + \mu_{t+1})] & \text{if} & \tilde{b}_{t+1} & < & b_{t+1} & \leq & \underline{b}_{t+1} \end{array}$$

The second line of this bond pricing schedule reflects the possibility of default in the future. In addition, if all consumers were risk-neutral with default options taking bond prices as given, the first-order condition for their bond holdings can be written as $q_t = (1+r)^{-1}E_t[(1+\mu_{t+1})^{-1}]$ where q_t denotes the market price for bond at period t and $\beta(1 + r) = 1$ holds. The difference from the canonical model of sovereign debt is that the Lagrange multiplier of the value-function constraint shown above is used to reflect the default risk, rather than calculating the default probability directly. This characterization of the magnitude of the default risk facilitates the use of interpolation methods of polynomials.

4.4 Piecewise Parameterized Expectations Approach

We now turn to the discussion of how to compute numerical solutions of equilibrium conditions by using the piecewise parameterized-expectations approach.

In order to show a practical example of this approach, the preferences of households are represented by a logarithmic utility function $u(c_t) = \log c_t$. In the first place, we begin with the range of $\tilde{b}_{t+1} < b_{t+1} \leq \underline{b}_{t+1}$ in which the pricing-function of bond reflects the possibility of default. The conditional expectation in the Euler equation is now parameterized by defining a new function as follows:

$$\kappa(b_t, y_t) \equiv \beta E_t[u'(c_{t+1})/(1 + \mu_{t+1})] - \gamma_t.$$

Given this parameterized conditional expectations, the Euler equation turns out to be

$$F_1(b_{t+1}, y_t)/c_t = \kappa(b_t, y_t).$$

In particular, this Euler equation is satisfied when the one-period flow budget-constraint of the representative household is

$$F(b_{t+1}, y_t) = \exp(\kappa(b_t, y_t)b_{t+1}) + b_t - y_t$$

and the consumption at period t of the representative household is

$$c_t = \exp(\kappa(b_t, y_t)b_{t+1}).$$

In this case, the value function under non-default option can be written as follows:

$$v^o(b_t, y_t) = \kappa(b_t, y_t)b_{t+1} + \beta E_t[v^o(b_{t+1}, y_{t+1})].$$

We now move onto the discussion of how to compute the conditional-expectation function. Given the value function and the consumption function shown above, the envelope condition leads to the following equation:

$$\kappa_1(b_t, y_t) = \exp(-\kappa(b_t, y_t)b_{t+1})(1 + \mu_t)^{-1}.$$

Combining the budget constraint and the risk-neutral bond pricing function together implies that the following condition holds:

$$\exp(\kappa(b_t, y_t)b_{t+1}) + b_t - y_t = \frac{b_{t+1}}{1 + r}E_t[(1 + \mu_{t+1})^{-1}]. \tag{4.1}$$

In particular, we note that this condition holds under the assumption that the government chooses the non-default option at period t. Moreover, with non-default option, the envelope condition turns out to be

$$\kappa_1(b_t, y_t) = \exp(-\kappa(b_t, y_t)b_{t+1}). \tag{4.2}$$

Figure 3: Debt-Revenue Function in the Model of Sovereign Debt
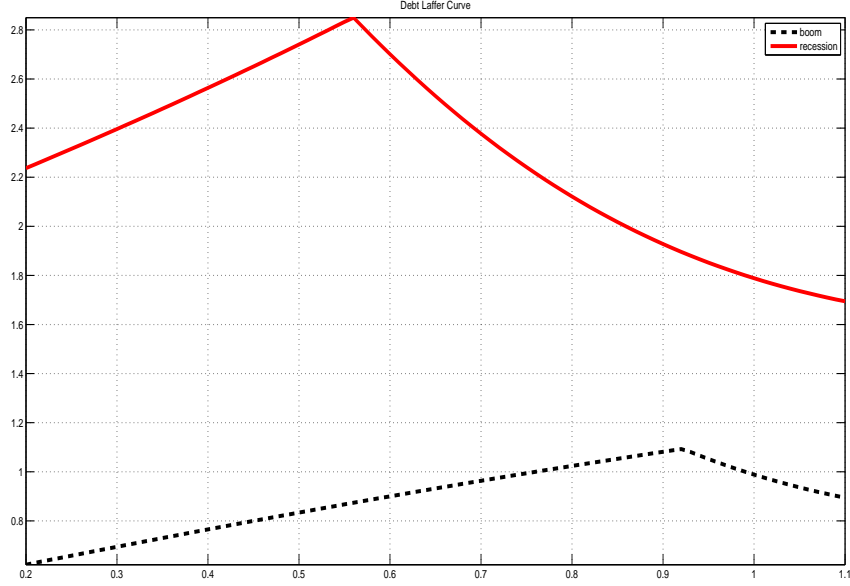
Note: The dotted line represents the debt revenue function at a higher level of output. The solid line corresponds to the debt revenue function at a lower level of output. The debt revenue is defined as the real price of bond times the real amount of debt that should be serviced in the next period.

We then solve (4.1) and (4.2) for both debt and conditional-expectation functions.

We now move onto the case where there is no default risk. In this case, the Euler equation gives a consumption policy function:

$$c_t = (1 + r)^{-1} \kappa^{nd}(b_t, y_t)^{-1}$$

where $\kappa^{nd}(b_t, y_t)$ represents the conditional expectations in the absence of default risk. The budget equation is then used to have a debt policy function

$$b_{t+1} = \kappa^{nd}(b_t, y_t)^{-1} + (1 + r)(b_t - y_t).$$

In this case, as is done before, we need to obtain a specific functional form of conditional expectation functions of the Euler equation. The conditional expectation is given by $\beta \int_{\tilde{y}'} c(b(b, y), y')^{-1} f(y', y) dy'$ where $b(b, y)$ is the debt policy function. The residual function is then defined as $R^{\text{pea}}(b, y; e) = \kappa^{nd}(b, y) - \beta \int_{\tilde{y}'} c(b(b, y), y')^{-1} f(y', y) dy'$. The policy function for the conditional expectation is approximated by the following Chebyshev polynomials: $\kappa^{nd}(b, y_i) \approx a'_{y_i} T(\varphi(b))$, where $T(x) = [T_0(x), T_1(x), \cdots, T_{N-1}(x)]$ and $\varphi(b) = 2(b - b_{\min})/(b_{\max} - b_{\min}) - 1$. An example of two states for the tradable output (y^h and y^l), the conditional expectations are parameterized by using the following functions: $\kappa(b, y_h) \approx a'_{y_h} T(\varphi(b))$ and $\kappa(b, y_h) \approx a'_{y_l} T(\varphi(b))$.

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Figure 3 plots the debt-revenue policy function. In this figure, the debt revenue rises initially and then declines until the government chooses the default option, leading to the Laffer-curve of debt-revenue.

5 Conclusion

The common feature of the two examples analyzed above is that there is a region of state variables in which equilibrium variables stay temporarily. Our contribution is to illustrate that the Euler equation approach can be used relatively efficiently for solving recent emerging-market macro models while the value function approach has been used to obtain numerical solutions to these models. Our future research is to make the Euler equation approach (employed in this paper) a more suitable for the analysis of emerging-market macro models.

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